Using identities to find $\sin 18^\circ$

By the double-angle formula for sine, we know that

$$\begin{aligned} \sin 72^\circ &= 2 \sin 36^\circ \cos 36^\circ \quad \text{and} \\ \sin 36^\circ &= 2 \sin 18^\circ \cos 18^\circ \ , \end{aligned}$$

so multiplying the two equations together gives

$$\sin 72^\circ \sin 36^\circ = 4 \sin 36^\circ \cos 36^\circ \sin 18^\circ \cos 18^\circ \ .$$

But $\sin 72^\circ = \cos(90^\circ - 72^\circ) = \cos 18^\circ$, and this substitution into the above equation allows some cancellation:

$$1\,\cancel{\cos 18^\circ}\,\cancel{\sin 36^\circ} = 4\,\cancel{\sin 36^\circ}\cos 36^\circ \sin 18^\circ\,\cancel{\cos 18^\circ}$$

Dividing both sides by 2 yields:

$$\frac{1}{2} = 2 \cos 36^\circ \sin 18^\circ \tag{1}$$

By the product-to-sum formula $2 \cos A \sin B = \sin(A+B) - \sin(A-B)$ with $A = 36^\circ$ and $B = 18^\circ$, we get

$$2 \cos 36^\circ \sin 18^\circ = \sin 54^\circ - \sin 18^\circ \ ,$$

which combined with equation (1) gives

$$\sin 54^\circ - \sin 18^\circ = \frac{1}{2} \quad \text{, and so}$$

$$\cos 36^\circ - \sin 18^\circ = \frac{1}{2} \tag{2}$$

since $\sin 54^\circ = \cos(90^\circ - 54^\circ) = \cos 36^\circ$. Also, by the factorization $a^2 - b^2 = (a+b)(a-b)$, we have

$$\begin{aligned} (\cos 36^\circ + \sin 18^\circ)^2 - (\cos 36^\circ - \sin 18^\circ)^2 &= \big(\cos 36^\circ + \sin 18^\circ + (\cos 36^\circ - \sin 18^\circ)\big)\big(\cos 36^\circ + \sin 18^\circ - (\cos 36^\circ - \sin 18^\circ)\big) \\ &= (2 \cos 36^\circ)(2 \sin 18^\circ) = 2\,(2 \cos 36^\circ \sin 18^\circ) \quad \text{, so} \end{aligned}$$

$$(\cos 36^\circ + \sin 18^\circ)^2 - (\cos 36^\circ - \sin 18^\circ)^2 = 1 \quad \text{by equation (1), and thus}$$

$$(\cos 36^\circ + \sin 18^\circ)^2 = 1 + (\cos 36^\circ - \sin 18^\circ)^2 = 1 + \left(\frac{1}{2}\right)^2 = \frac{5}{4} \quad \text{by equation (2), so}$$

$$\cos 36^\circ + \sin 18^\circ = \frac{\sqrt{5}}{2} \ . \tag{3}$$

Since $\cos 36^\circ + \sin 18^\circ - (\cos 36^\circ - \sin 18^\circ) = 2 \sin 18^\circ$, subtracting equation (2) from equation (3) gives

$$2 \sin 18^\circ = \frac{\sqrt{5}}{2} - \frac{1}{2} \quad \text{, and so}$$

$$\boxed{\sin 18^\circ = \frac{\sqrt{5} - 1}{4}} \ .$$

Note that this allows us to find $\cos 18^\circ$:

$$\cos^2 18^\circ = 1 - \sin^2 18^\circ = 1 - \left(\frac{\sqrt{5} - 1}{4}\right)^2 = \frac{10 + 2\sqrt{5}}{16} \quad \Rightarrow \quad \cos 18^\circ = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$$